Graphs and Combinatorics 2, 191–200 (1986)

Graphs and Combinatorics

Spanning Tree Formulas and Chebyshev Polynomials

F.T. Boesch[1]* and H. Prodinger[2]

[1] Computer Science Department, Stevens Institute of Technology, Hoboken, NJ, USA
[2] Institut für Algebra und Diskrete Mathematik, Technische Universität, Wien, Austria

Abstract. The Kirchhoff Matrix Tree Theorem provides an efficient algorithm for determining $t(G)$, the number of spanning trees of any graph G, in terms of a determinant. However for many special classes of graphs, one can avoid the evaluation of a determinant, as there are simple, explicit formulas that give the value of $t(G)$. In this work we show that many of these formulas can be simply derived from known properties of Chebyshev polynomials. This is demonstrated for wheels, fans, ladders, Moebius ladders, and squares of cycles. The method is then used to derive a new spanning tree formula for the complete prism $R_n(m) = K_m \times C_n$. It is shown that

$$t(R_n(m)) = \frac{n}{m}2^{m-1}\left[T_n\left(1+\frac{m}{2}\right)_{-1}\right]^{m-1}$$

where $T_n(x)$ is the n^{th} order Chebyshev polynomial of the first kind.

1. Introduction

Herein we give derivations of simple explicit formulas giving the number of spanning trees for certain special classes of graphs. We follow the notation and terminology of the book [7] by Harary. For any graph or multigraph G, we denote by $t(G)$ the total number of spanning trees of G, and we let $A(G)$ or A denote the adjacency matrix of G. The celebrated Kirchhoff Matrix Tree Theorem [10], cf [7], states that if D is the diagonal matrix of the degrees of G, then the Kirchhoff matrix H defined as $H = D - A$ has all of its co-factors equal to $t(G)$. Other methods for calculating $t(G)$ are as follows. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p$ denote the eigenvalues of the H matrix of a p point graph. Then it is easily shown that $\mu_p = 0$. Furthermore, Kel'mans and Chelnokov [9] have shown that

$$t(G)\frac{1}{p}\prod_{k=1}^{p-1}\mu_k. \tag{1}$$

From this it is a simple matter to derive the result of Sachs [14], cf [15], which states that if G is regular of degree r then

---

* The work of this author was supported under NSF Grant ECS-8100652.

$$t(G) = \frac{1}{p} \prod_{k=1}^{p-1} (r - \lambda_k) \tag{2}$$

where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p = r$ are the eigenvalues of the adjacency matrix A. Finally we state Temperley's result [21], cf [3],

$$t(G) = \frac{1}{p^2} \det(H + J) \tag{3}$$

where J is the $p \times p$ matrix all of whose elements are unity. Notice that this last formula has the advantage of expressing $t(G)$ directly as a determinant rather than in terms of cofactors or eigenvalues.

All of the above results lead to efficient algorithms for calculating $t(G)$ for any graph G. However the required calculations can be tedious and there is certainly an advantage to having a simple explicit formula for special classes of graphs. For example, it is well-known that for the complete graph K_p, $t(K_p) = p^{p-2}$, and this follows immediately from (3). However, equations (1), (2), and (3) do not always yield such simple derivations of $t(G)$ formulas. For example, $A(C_p)$ is a circulant matrix, and explicit formulas for a circulant's eigenvalues can be found in Marcus and Minc [12]. Now although it is obvious that $t(C_p)$ is p, the application of formula (2) does not easily demonstrate this fact, as the eigenvalue approach would require a verification of the somewhat surprising formula

$$t(C_p) = \frac{1}{p} \prod_{k=1}^{p-1} [2 - 2\cos(2\pi k/p)] = \frac{1}{p} \prod_{k=1}^{p-1} [4\sin^2(\pi k/p)] = p. \tag{4}$$

In this work we shall show that simple explicit formulas can be easily derived for some classes of graphs by using known properties of the Chebyshev polynomials. We derive a new formula for the complete prism $R_n(m) = K_m \times C_n$. We then show that similar approaches lead to easy proofs of the known formulas for: the wheel $W_n = C_{n-1} + K_1$, the Moebius ladder M_n, the fan $F_n = P_{n-1} + K_1$, the square of a cycle C_p^2, and the ladder $L_n = K_2 \times P_n$. To this end we need to review some known properties. First we note that it is not difficult to show that if x is an edge of G, then the number of trees which contain x is $t(G/x)$ where G/x denotes the graph obtained by coalescing the endpoints of the edge x. This result together with the inclusion-exclusion principle yields the result, which is apparently due to Feussner [6], cf. Moon [13] that

$$t(G) = t(G - x) + t(G/x). \tag{5}$$

From this we see that if G_s denotes the graph that results from subdividing an edge z of G, then

$$t(G_s) = t(G/z) + 2t(G - z) = t(G) + t(G - z). \tag{6}$$

Likewise if G_p denotes the result of adding an edge in parallel with an edge z of G, then

$$t(G_p) = t(G) + t(G/z). \tag{7}$$

At this point, we shall digress to review some basic properties of Chebyshev polynomials. These will be used together with the above results to derive the new formulas in the subsequent section.

2. Some Properties of Chebyshev Polynomials and Fibonacci Numbers

Our primary reference for properties of the Chebyshev polynomials is Snyder [20]. First we define an $n \times n$ matrix

$$A_n(x) = \begin{bmatrix} 2x & -1 & 0 & 0 & & & \\ -1 & 2x & -1 & 0 & & & \\ 0 & -1 & 2x & -1 & & & \\ 0 & 0 & -1 & \cdot & & & 0 \\ & & & & \cdot & & \\ & & & & & \cdot & -1 \\ & & & & 0 & -1 & 2x \end{bmatrix}$$

where all elements not shown are assumed to be zero. Further we recall that the Chebyshev polynomials of the first kind are defined by

$$T_n(x) = \cos(n \arccos x). \tag{8}$$

The Chebyshev polynomials of the second kind are defined by

$$U_{n-1}(x) = \frac{1}{n}\frac{d}{dx}T_n(x) = \frac{\sin(n \arccos x)}{\sin(\arccos x)}. \tag{9}$$

It is easily verified that

$$U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0. \tag{10}$$

It can then be shown from this recursion that by expanding $\det A_m(x)$ one gets

$$\det A_m(x) = U_m(x) \quad \text{for } m \geq 1. \tag{11}$$

Furthermore by using standard methods for solving the recursion (10), one obtains the explicit formula

$$U_m(x) = \frac{1}{2\sqrt{x^2-1}}\left[\left(x + \sqrt{x^2-1}\right)^{m+1} - \left(x - \sqrt{x^2-1}\right)^{m+1}\right]. \tag{12}$$

Another interesting fact follows by comparing (12) with the well-known closed from formula for the Fibonacci numbers f_m

$$f_m = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^m - \left(\frac{1-\sqrt{5}}{2}\right)^m\right], \tag{13}$$

namely

$$U_{m-1}(3/2) = f_{2m} = \frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^m - \left(\frac{3-\sqrt{5}}{2}\right)^m\right]. \tag{14}$$

In the next section, we explore the use of the above facts to derive two new tree formulas.

3. The Formula for Complete Prisms

Before on embarking on the proof, we digress slightly to observe some further properties of Chebyshev polynomials. The definition of $U_n(x)$ easily yields its zeros, and one verfies

$$U_{m-1}(x) = 2^{m-1} \prod_{k=1}^{m-1} [x - \cos(k\pi/m)]. \tag{15}$$

Further we note that

$$U_{m-1}(-x) = (-1)^{m-1} U_{m-1}(x). \tag{16}$$

These two results yield the formula

$$U_{m-1}^2(x) = 4^{m-1} \prod_{k=1}^{m-1} [x^2 - \cos^2(k\pi/m)]. \tag{17}$$

Furthermore one can show that

$$U_{m-1}^2(x) = \frac{1}{2(1-x^2)}[1 - T_{2m}(x)] = \frac{1}{2(1-x^2)}[1 - T_m(2x^2 - 1)], \tag{18}$$

and

$$T_n(x) = \frac{1}{2}\left[\left(x + \sqrt{x^2 - 1}\right)^n + \left(x - \sqrt{x^2 - 1}\right)^n\right]. \tag{19}$$

Theorem 1. *If $R_n(m) = K_m \times C_n$ denotes the complete prism then*

$$t(R_n(m)) = \frac{n2^{m-1}}{m}\left[T_n\left(1 + \frac{m}{2}\right) - 1\right]^{m-1}$$
$$= \frac{n}{m}\left[\left(\frac{m + 2 + \sqrt{m^2 + 4m}}{2}\right)^n + \left(\frac{m + 2 - \sqrt{m^2 + 4m}}{2}\right)^n - 2\right]^{m-1}$$

Proof. First we observe that the adjacency matrix of $G \times H$ is the Kronecker sum, cf [2], of the adjacency matrices of G and H. It is known that the eigenvalues of the Kronecker sum of two matrices are all possible sums of the individual matrices [2]. Now $A(K_m)$ has $m - 1$ eigenvalues equal to -1 and one equal to $m - 1$. Furthermore $A(C_n)$ has $2\cos(2\pi k/n)$ for $0 \leq k \leq n - 1$ as its eigenvalues. Hence $A(K_m \times C_n)$ has as its eigenvalues

$$m - 1 + 2\cos(2\pi k/n) \quad \text{for } 0 \leq k \leq n - 1$$

and the following, each of order $m - 1$

$$-1 + 2\cos(2\pi k/n) \quad \text{for } 0 \leq k \leq n - 1.$$

The degree of $K_m \times C_n$ is $m - 1 + 2$. Hence by (2)

$$t(K_m \times C_n) = \frac{1}{nm}\left[\prod_{k=0}^{n-1} [m + 2 - 2\cos(2\pi k/n)]\right]^{m-1} \prod_{k=1}^{n-1} [2 - 2\cos(2\pi k/n)]. \tag{20}$$

Now the second term in equation (20) is easily seen via (4) to be $nt(C_n)$. Some simple trigonometric manipulation then yields

$$t(K_m \times C_n) = m^{m-2}n\left[\prod_{k=1}^{n-1}[m + 4 - 4\cos^2(\pi k/n)]\right]^{m-1}. \tag{21}$$

Thus by (17) we have

$$t(K_m \times C_n) = nm^{m-2}\left[U_{n-1}^2\left(\sqrt{\frac{m+4}{4}}\right)\right]^{m-1}.$$

By (18) and (19) we have the conclusion. □

An interesting special case arises for $m = 2$, which is the prism $K_2 \times C_n$. Hence

$$t(K_2 \times C_n) = \frac{n}{2}[(2 + \sqrt{3})^n + (2 - \sqrt{3})^n - 2],$$

a result which was given in [4].

In the next section, we shall see that this cyclic prism (sometimes called a cycle permutation graph) has a tree formula quite similar to the formula for $t(M_n)$, the Moebius ladder tree formula.

4. Simple Derivations of Some Known Tree Formulas

The wheel $W_n = K_1 + C_{n-1}$ has been studied extensively. An explicit formula for $t(W_n)$ was derived by Sedláček [17]. We shall give a simple proof of that result subsequently. Now we merely note that, if x is any edge incident at the point of degree n in W_{n+1}, then the multigraph W_{n+1}/x has $\det A_{n-1}(3/2)$ as a cofactor of its Kirchhoff Matrix. Hence if we call such an edge x a spoke, then for any spoke x

$$t(W_{n+1}/x) = U_{n-1}(3/2) = f_{2n}. \tag{22}$$

The next theorem results from exploring a consequence of this observation; the formula was originally obtained by Hilton [8]. However, the proof here is quite simple.

Theorem 2. *Define a fan F_{m+1} as $P_m + K_1$, then*

$$t(F_{m+1}) = U_{m-1}(3/2) = f_{2m} = \frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^m - \left(\frac{3-\sqrt{5}}{2}\right)^m\right].$$

Proof. Let any edge y of W_{n+1} that is not incident at the point of degree n be called a rim. Then $W_{n+1} - y = P_n + K_1 = F_{n+1}$. Furthermore we know that $W_{n+1}/x = U_{n-1}(3/2)$ when x is a spoke. Thus we need only to show that $t(W_{n+1} - y) = t(W_{n+1}/x)$. This can be verified by induction as follows. It is certainly true for $n = 3$. Next observe that $W_{n+1} - x$ results from subdividing a rim and W_{n+1}/y results from adding an edge in parallel with a spoke. Thus by (6) and (7)

$$t(W_{n+1} - x) = t(W_n) + t(W_n - y)$$

$$t(W_{n+1}/y) = t(W_n) + t(W_n/x)$$

Hence if

$$t(W_n - y) = t(W_n/x)$$

then

$$t(W_{n+1} - x) = t(W_{n+1}/y).$$

However

$$t(W_{n+1}) = t(W_{n+1} - x) + t(W_{n+1}/x)$$
$$= t(W_{n+1} - y) + t(W_{n+1}/y).$$

Hence it follows that

$$t(W_{n+1} - y) = t(W_{n+1}/x).$$ □

Now we show that Sedláček's formula for $t(W_n)$ [15] follows immediately from a similar approach.

Theorem 3. *Let $W_n = K_1 + C_{n-1}$ denotes an an point wheel. Then*

$$t(W_n) = 2[T_{n-1}(3/2) - 1] = \left(\frac{3+\sqrt{5}}{2}\right)^{n-1} + \left(\frac{3-\sqrt{5}}{2}\right)^{n-1} - 2.$$

Proof. Following [12] we use $H(i/j)$ to denote the submatrix obtained by deleting row i and column j of a matrix H. Thus if the Kirchhoff matrix of W_n is H, where the last row and column correspond to the point of degree $n-1$, we have

$$H(n/n) = 3I_{n-1} - A(C_{n-1})$$

where I_k is the identity matrix of order k. Thus if ν_k denotes the eigenvalues of H, we have

$$t(W_n) = \det H(n/n) = \prod_{i=0}^{n-2} \nu_k = \prod_{i=0}^{n-2} [3 - \lambda_i(A(C_{n-1}))].$$

Hence by (4) we have

$$\begin{aligned} t(W_n) &= \prod_{i=0}^{n-2} [3 - 2\cos(2\pi i/(n-1))] \\ &= \prod_{i=1}^{n-2} [5 - 4\cos^2(\pi i(n-1))] \\ &= U_{n-2}^2(\sqrt{5}/2) \qquad \text{by (17)} \\ &= 2[T_{n-1}(3/2) - 1] \qquad \text{by (18)} \end{aligned}$$

Again the explicit formula follows from (19). □

Next we give a simple derivation of the formula for $t(C_n^2)$. The formula was originally conjectured by Bedrosian and subsequently proven by Kleitman and Golden [11]. The same formula was also conjectured by Boesch and Wang [5] (without knowledge of [11]). A different proof of the formula was given by Baron, Boesch, Prodinger, Tichy, and Wang [1]. The proof given below is simpler than either of the previous proofs.

Theorem 4. $t(C_n^2) = nf_n^2$ *for* $n \geq 5$

Proof. Since $A(C_n^2)$ is also a circulant, we can use the explicit formulas [12] for its eigenvalues as

$$\lambda_i = 2\cos(2\pi i/n) + 2\cos(4\pi i/n) \quad (\text{for } 1 \leq i \leq n),$$

where $n \geq 5$ is assumed so that C_n^2 is a graph. Substituting in (2) and using some simple trigonometric manipulation we get

$$t(C_n^2)\frac{1}{n}\prod_{k=1}^{n-1}[1 + 4\cos^2(\pi k/n)][4\sin^2(\pi k/n)].$$

Recognizing that the second term in this expression is the same as (4), we see that the proof is equivalent to showing that

$$f_n^2 = \prod_{k=1}^{n-1}[1 + 4\cos^2(\pi k/n)].$$

However by (17) we obtain

$$\prod_{k=1}^{n-1}[1 + 4\cos^2(\pi k/n)] = (-1)^{n-1}U_{n-1}^2(1/2i)$$

where $i = \sqrt{-1}$.

We now note that

$$f_n = (-1)^{(n-1)/2}U_{n-1}(1/2i)$$

by using the explicit formula (12) for $U_{n-1}(1/2i)$ and composing the result with the explicit formula (13) for f_n. Thus the proof is completed. □

Next we consider the Moebius ladder M_n which may be defined as follows. Consider the cycle C_{2n} and join every pair of points that are distance in apart on C_{2n} by an edge. The reason for the name is that this graph can be formed from $K_2 \times P_n$ by adding edge from the first point on one copy of P_n to the last point on the second copy of P_n and an edge from the first point on the second copy to the last point on the first copy.

The next formula is stated without proof in Biggs [3] and Moon [13]; the result is due to Sedlácěk [19].

Theorem 5.

$$\begin{aligned} t(M_n) &= \frac{n}{2}[(2 + \sqrt{3})^n + (2 - \sqrt{3})^n + 2] \\ &= n[T_n(2) + 1] \end{aligned}$$

Proof. Again using the eigenvalue formula for a circulant [12] and substituting into (2) we get

$$t(M_n) = \frac{1}{2n}\prod_{k=1}^{n}[4 - 2\cos[(2k-1)\pi/n)]\prod_{k=1}^{n-1}[2 - 2\cos(2k\pi/n)].$$

As before we use (4) to obtain

$$t(M_n) = \frac{n}{2} \prod_{k=1}^{n} [4 - 2\cos((2k-1)\pi/n)].$$

Now recall that in Theorem 1 we had that

$$t(K_2 \times C_n) = n \prod_{k=1}^{n-1} [4 - 2\cos(2\pi k/n)].$$

Hence

$$t(M_n)t(K_2 \times C_n) = \frac{n^2}{2} \prod_{k=1}^{2n-1} [4 - 2\cos(k\pi/n)],$$

which can be reduced to

$$3n^2 4^{n-1} \prod_{k=1}^{n-1} [4 - \cos^2(k\pi/n)].$$

Now invoking (17), (18), and (19) we get

$$t(M_n)t(K_2 \times C_n) = \frac{n^2}{4}[(2+\sqrt{3})^{2n} + (2-\sqrt{3})^{2n} - 2].$$

Using

$$t(K_2 \times C_n) = \frac{n}{2}[(2+\sqrt{3})^n + (2-\sqrt{3})^n - 2] = n[T_n(2) - 1],$$

the desired result is obtained. □

We complete this section by noting that a similar procedure gives a formula for $t(L_n)$, where L_n is the ladder $K_2 \times P_n$. This result, which is also due to Sedlácěk [18], is stated here as Theorem 6.

Theorem 6. *Let L_n denote $K_2 \times P_n$, then*

$$t(L_n) = U_{n-1}(2) = \frac{1}{2\sqrt{3}}[(2+\sqrt{3})^n - (2-\sqrt{3})^n].$$

Proof. First we digress to define the $n \times n$ matrix

$$B_n(x) = \begin{bmatrix} x & -1 & & & & \\ -1 & (1+x) & -1 & & & \\ 0 & -1 & (1+x) & -1 & & \\ & & & \ddots & & \\ & & & & (1+x) & 1 \\ & & & & -1 & x \end{bmatrix}$$

where all elements not shown are assumed to be zero. Then it is easily shown that

$$\det B_n(x) = (x-1)U_{n-1}\left(\frac{1+x}{2}\right).$$

Now the characteristic polynomial of the Kirchhoff matrix corresponding to L_n is

$$\det\left[\begin{array}{c|c} B_n(2-\mu) & -I_n \\ \hline -I_n & B_n(2-\mu) \end{array}\right]$$

This is easily seen to be equal to

$$\begin{aligned} &\det[B_n(2-\mu) - I_n]\det[B_n(2-\mu) + I_n] \\ &= \mu(\mu-2)U_{n-1}\left(\frac{2-\mu}{2}\right)U_{n-1}\left(\frac{4-\mu}{2}\right). \end{aligned}$$

Using the fact that the roots of $U_{n-1}(x)$ are $\cos(k\pi/n)$ for $1 \leq k \leq n-1$, we obtain the eigenvalues μ of the H matrix as

$$0;\ 2;\ 2[1 - \cos(k\pi/n)] \quad \text{for } 1 \leq k \leq n-1; \text{ and}$$
$$2[2 - \cos(k\pi/n)] \quad \text{for } 1 \leq k \leq n-1.$$

Hence applying (1) together with (4) and (15) we get,

$$\begin{aligned} t(L_n) &= \frac{2}{2n}\prod_{k=1}^{n-1}[2 - 2\cos(k\pi/n)]\prod_{k=1}^{n-1}2[2-\cos(k\pi/n)] \\ &= \frac{1}{n}U_{n-1}(1)U_{n-1}(2) = U_{n-1}(2) \end{aligned}$$

Again the explicit formula follows immediately from (12). □

4. Conclusions

By noting a connection between the Kirchhoff matrix and known properties of Chebyshev polynomials, we are led to simple proofs of explicit $t(G)$ formulas for certain classes of graphs. The derivation of explicit tree formulas via Chebyshev polynomials does not appear to have been noted previously. Furthermore it leads to the following new compact version of known formulas:

$$\begin{aligned} t(L_n) &= U_{n-1}(2), \\ t(W_n) &= 2[T_{n-1}(3/2) - 1], \\ t(M_n) &= n[T_n(2) + 1], \\ t(C_n^2) &= nf_n^2, \quad \text{and} \\ t(F_{m+1}) &= f_{2m}. \end{aligned}$$

We have also shown that there is a simple formula for $t(R_n(m))$. In the latter case it could be said that there already existed a formula as the eigenvalues of $A(R_n(m))$ were known [16], and equation (2) was known. Thus equation (21) given in the proof of Theorem 1 follows by merely manipulating the results developed in [16]. However, the formula given herein is considerably simpler than equation (21).

Acknowledgment. The authors wish to thank Professor Philippe Flajolet of INRIA, France for discussions which led us to the approach given here.

References

1. Baron, G., Boesch, F., Prodinger, H., Tichy, R., Wang, J.: The number of spanning trees in the square of a cycle. Fibonacci Q. (to appear)
2. Bellman, R.: Introduction to Matrix Analysis. New York: McGraw Hill 1970
3. Biggs, N.: Algebraic Graph Theory. London: Cambridge University Press 1974
4. Boesch, F., Bogdanowicz, Z.: The number of spanning trees in a prism. Stevens Institute Computer Science Report 8405 (1984)
5. Boesch, F.T., Wang, J.F.: A conjecture on the number of spanning trees in the square of a cycle. In: Notes from New York Graph Theory Day V, p. 16. New York: New York Academy Sciences 1982
6. Feussner, W.: Zur Berechnung der Stromsträrke in netzförmigen Leitern. Ann. Phys. **15**, 385–394 (1904)
7. Harry, F.: Graph Theory. Reading: Addison-Wesley 1969
8. Hilton, A.J.W.: Spanning trees and Fibonacci and Lucas numbers. Fibonacci Q. **12**, 259–262 (1974)
9. Kel'mans, A.K., Chelnokov, V.M.: A certain polynomial of a graph and graphs with an extremal number of trees. J. Comb. Theory (B) **16**, 197–214 (1974)
10. Kirchhoff, G.: Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme geführt wird. Ann. Phys. Chem. **72**, 497–508 (1847)
11. Kleitman, D.J., Golden, B.: Counting trees in a certain class of graphs. Amer. Math. Mon. **82**, 40–44 (1975)
12. Marcus, M., Minc, H.: A Survey of Matrix Theory and Matrix Inequalities. Boston: Allyn and Bacon 1964
13. Moon, J.W. Counting Labelled Trees. Canadian Mathematical Congress Monograph 1. London: William Clowes & Sons 1970
14. Sachs, H.: Über selbstkomplementäre Graphen. Publ. Math. Debrecen **9**, 270–288 (1962)
15. Schwenk, A., Wilson, R.: On the eigenvalues of a graph. In: Selected Topics in Graph Theory, edided by Beineke, Wilson, pp. 307–336. New York: Academic Press 1978
16. Schwenk, A.: Computing the characteristic polynomial of a graph. In: Graphs and Combinatorics, Lecture Notes in Mathematics **406**, pp. 153–172. Berlin-Heidelberg-New York: Springer-Verlag 1974
17. Sedláček, J.: Lucas numbers in graph theory. In: Mathematics-Geometry and Graph Theory (in Czechoslovak), pp. 111–115. Prague: Univ. Karlova 1970. See also Sedláček, J.: Ungerichtete Graphen und ihre Gerüste. In: Beiträge zur Graphen Theorie, edited by (H. Sachs, H.-J. Vosz, H. Walther, pp. 143–146. Leipzig: Teubner 1968
18. Sedláček, J.: On the spanning trees of finite graphs. Cas. Pestovani Mat. **94**, 217–221 (1969)
19. Sedláček, J.: On the skeletons of a graph or digraph. In: Combinatorial Structures and their Applications, editded by R. Guy, M. Hanani, N. Saver, J. Schonheim, pp. 387–391. New York: Gordon and Breach 1970
20. Snyder, M.: Chebyshev Methods in Numerical Analysis. Englewood Cliffs: Prentice-Hall 1966
21. Temperley, H.: On the mutual cancellation of cluster integrals in Mayer's fugacity series. Proc. Phys. Soc. Lond. Ser. A **83**, 3–16 (1964)

Received: August 1, 1985
Revised: February 1, 1986